常微分方程的求解

① 可分离变量的方程 $y' = f(x)g(y)$

② 可化为分离变量的方程

1° $y' = f(ax+by+c)$ $(b \neq 0)$

变换 $z = ax+by+c$

$$\Rightarrow \frac{dz}{dx} = a + bf(z)$$

2° 齐次方程 $y' = f(\frac{y}{x})$

变换 $u = \frac{y}{x}$

$$\Rightarrow \frac{du}{dx} = \frac{f(u)-u}{x}$$

3° 可化为齐次方程的方程 $y' = f(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2})$

1) $(a_1, b_1) \times (a_2, b_2) \neq \vec{0}$

变换 $\begin{cases} a_1u + b_1v = a_1x + b_1y + c_1 \\ a_2u + b_2v = a_2x + b_2y + c_2 \end{cases}$

$$\Rightarrow \frac{dv}{du} = f(\frac{a_1u+b_1v}{a_2u+b_2v})$$

2) $(a_1, b_1) \times (a_2, b_2) = \vec{0}$

变换 $z = a_1x + b_1y$

$$\Rightarrow \frac{dz}{dx} = a_1 + b_1 f(\frac{z+c_1}{\lambda z + c_2})$$

③ 一阶线性方程 $y' + p(x)y = q(x)$

1) 齐次方程 $y' + p(x)y = 0$

$$y = Ce^{-\int p(x)dx}$$

2) 非齐次方程

$$y = e^{-\int p(x)dx}\left(C + \int q(x)e^{\int p(x)dx}dx\right)$$

或 $y = e^{-\int_{x_0}^{x} p(\xi)d\xi}\left(y_0 + \int_{x_0}^{x} q(\xi)e^{\int_{\xi_0}^{\xi} p(t)dt}d\xi\right)$ $(y(x_0) = y_0)$

④ 伯努利方程 $y' + p(x)y = q(x)y^n \quad (n \neq 0, 1)$

变换 $z = y^{1-n}$

$$\Rightarrow \frac{dz}{dx} + [(1-n)p(x)]z = [(1-n)q(x)]$$

⑤ 全微分方程

1° $dU(x,y) = 0$

2° 用积分因子将方程化为全微分方程

$$\mu M(x,y)dx + \mu N(x,y)dy = dU(x,y) = 0$$

~~⑥ 里卡蒂方程 $y' = P(x)y^2 + Q(x)y + R(x)$~~

⑥ 部分里卡蒂方程 $y' + by^2 = ax^m$ $\quad (m = -2, \frac{-4k}{2k+1}, \frac{-4k}{2k+1})\ (k = 1, 2, 3, \cdots)$

1° $m = -2$

变换 $z = xy$

⑦ y'的n次多项式 $A_n(x,y)y'^n + A_{n-1}(x,y)y'^{n-1} + \cdots + A_0(x,y) = 0.$

因式分解

⑧ $F(y') = 0$

若$F(y')=0$至少有一实根$y'=k$.

则 $y = kx + C$ 是一个解.

$\therefore k = \frac{y-C}{x}$

$\therefore F(\frac{y-C}{x}) = 0$，即为积分。

⑨ $F(x, y') = 0$

设 $x = \varphi(t)$, $p = \psi(t)$

则 $dy = p\,dx = \psi(t)\varphi'(t)dt$

$$\therefore \begin{cases} x = \varphi(t) \\ y = \int \psi(t)\varphi'(t)dt + C \end{cases}$$

⑩ $F(y, y') = 0$

设 $y = g(t)$，$p = h(t)$.

则 $dx = \frac{1}{p}dy = \frac{g'(t)}{h(t)}dt$

$$\therefore \begin{cases} x = \int \frac{g'(t)}{h(t)}dt + C \\ y = g(t) \end{cases}$$

⑪ $y = f(x, p)$

$$p = f'_x(x, p) + f'_p(x, p)\frac{dp}{dx}$$

1° 若有解 $p = u(x, c)$

则 $y = f(x, u(x, c))$

2° 若有解 $x = v(p, c)$

则 $\begin{cases} x = v(p, c) \\ y = f(v(p, c), p) \end{cases}$